BROJNI REDOVI

106. Ispitati konvergenciju redova

a) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^3+1}}{n}$

$\rightarrow$ opšti član je $a_n = \frac{\sqrt[3]{n^3+1}}{n}$

JORDI LABANDA

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sqrt[3]{n^3(1+\frac{1}{n^3})}}{n} = \lim_{n\to\infty} \frac{n\sqrt[3]{1+\frac{1}{n^3}}}{n} = 1 \neq 0$$

$\rightarrow$ red $\sum_{n=1}^{\infty} a_n$ ne konvergira, tj. divergira.

$\rightarrow$ Da bi red konvergirao $\rightarrow$ njegov opšti član mora da teži nuli

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b) $\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$

$\rightarrow$ opšti član $a_n = \frac{n+1}{2n+3}$

$\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} \neq 0 \Rightarrow$ red $\sum_{n=1}^{\infty} \frac{n+1}{2n+3}$ divergira

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c) $\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$ $\rightarrow$ niz parcijalnih suma ne konvergira $\rightarrow$ ne konvergira ni bojni red

$a_n = \ln \frac{n}{n+1} = \ln n - \ln(n+1)$

$S_n = a_1 + a_2 + \dots + a_n = \ln 1 - \cancel{\ln 2} + \cancel{\ln 2} - \cancel{\ln 3} + \dots + \cancel{\ln n} - \ln(n+1)$

$= \underbrace{\ln 1}_{0} - \ln(n+1) = -\ln(n+1)$

$\lim_{n\to\infty} S_n = -\lim_{n\to\infty} \ln(n+1) = -\infty \rightarrow$

Niz $(S_n)_{n\in\mathbb{N}}$ ne konvergira, pa ne konvergira ni red $\sum_{n=1}^{\infty} a_n$

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d) $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ $\quad a_n = \ln \frac{n^2+1}{2n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(2+\frac{1}{n^2}\right)}\right) = \ln\frac{1}{2} \neq 0$$

$\Rightarrow$ red $\sum_{n=1}^{\infty} a_n$ ne konvergira

(107.) Naći sumu reda $\sum_{n=1}^{\infty} a_n$, gdje je a_n

$$a_n = \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}$$

$\Rightarrow$ posmatra se niz parcijalnih suma

$$S_n = a_1 + a_2 + \cdots + a_n$$

$$S_n = \left(\sqrt{3} - 2\sqrt{2} + \sqrt{1}\right) + \left(\sqrt{4} - 2\sqrt{3} + \sqrt{2}\right) + \left(\sqrt{5} - 2\sqrt{4} + \sqrt{3}\right) + \cdots$$

$$+ \left(\sqrt{n} - 2\sqrt{n-1} + \sqrt{n-2}\right) + \left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1}\right) +$$

$$+ \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n}$$

$$S_n = 1 - \sqrt{2} - \sqrt{n+1} + \sqrt{n+2}$$

racionališemo

gr. vr. niza parc. suma

$$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \sqrt{2} + \left(\left(\sqrt{n+2} - \sqrt{n+1}\right) \cdot \frac{\sqrt{n+2} + \sqrt{n+1}}{\sqrt{n+2} + \sqrt{n+1}}\right)\right) =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+2} + \sqrt{n+1}}\right) = 1 - \sqrt{2}$$

$\Rightarrow$ NPS S_n konvergira pa konvergira i red $\sum_{n=1}^{\infty} a_n$

(108) Naći sumu reda $\sum_{n=1}^{\infty} \frac{1}{(3n-2)(3n+1)}$

$$a_n = \frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1}$$

$$A = \frac{1}{3}; \quad B = -\frac{1}{3}$$

$$a_n = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)$$

$$\Rightarrow S_n = a_1 + a_2 + \dots + a_n =$$

$$= \frac{1}{3}\left(\frac{1}{1} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \dots + \cancel{\frac{1}{3n-2}} - \frac{1}{3n+1}\right) =$$

$$= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{1}{3}$$

$\rightarrow$ Niz parcijalnih suma (S_n) konvergira $\rightarrow$

$\rightarrow$ konvergira i red $\sum_{n=1}^{\infty} a_n$.

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(109) Naći sumu reda $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{2^{n-1}} = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^{n-1}$$

$$\rightarrow a_n = \left(-\frac{1}{2}\right)^{n-1}; \quad S_n = a_1 + a_2 + \dots + a_n$$

$$S_n = 1 - \frac{1}{2} + \frac{1}{4} + \dots + \left(-\frac{1}{2}\right)^{n-1} = 1 \cdot \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} =$$

$$= \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$$

$$\lim_{n\to\infty} \frac{2}{3}\left(1 - \underbrace{\left(-\frac{1}{2}\right)^n}_{0}\right) = \frac{2}{3} \rightarrow \text{niz } S_n \text{ konvergira}$$

pa konvergira i red $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^{n-1}}$

(110) Ispitati konvergenciju i naći sumu reda $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

$$a_n = \frac{e^n}{3^{n-1}} = e\left(\frac{e}{3}\right)^{n-1}$$

$\rightarrow$ Ovo je geometrijski red $\rightarrow e, e\frac{e}{3}, e\left(\frac{e}{3}\right)^2, \ldots$

$\rightarrow q = \frac{e}{3} \rightarrow |q| < 1 \rightarrow$ Dati geometrijski red konvergira.

(111) Ispitati konvergenciju reda $\sum_{n=1}^{\infty}\left(\frac{1}{e^n} + \frac{1}{n(n+1)}\right) =$

$$= \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$\rightarrow$ opšti član $a_n = \frac{1}{e^n} + \frac{1}{n(n+1)}$

$\rightarrow$ Neka je $b_n = \frac{1}{e^n}$ i $c_n = \frac{1}{n(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{e^n}$ konvergira kao geometrijski red kod kojeg je $q = \frac{1}{e} < 1$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$c_n = -\frac{1}{n+1} + \frac{1}{n} \rightarrow \frac{1}{n} - \frac{1}{n+1}$$

$$S_n = c_1 + c_2 + \ldots + c_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \ldots + \cancel{\frac{1}{n}} - \frac{1}{n+1} =$$

$$= 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n \left(1 - \cancel{\frac{1}{n+1}}\right) = 1 \rightarrow$ niz S_n konv $\rightarrow$

$\rightarrow$ konv. i red $\sum_{n=1}^{\infty} c_n$

$$\left.\begin{array}{l}\sum_{n=1}^{\infty} b_n \text{ konv} \\ \sum_{n=1}^{\infty} c_n \text{ konv}\end{array}\right\} \Rightarrow \text{konv. i red } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n + c_n)$$

Hiperharmonijski red $\sum_{n=1}^{\infty} \frac{1}{n^p}$ konv. za $p > 1$, div za $p \leq 1$

(112.) Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$

$a_n = \frac{3}{5^n} + \frac{2}{n}$, $b_n = \frac{3}{5^n}$, $c_n = \frac{2}{n}$

$\sum_{n=1}^{\infty} b_n = 3 \cdot \sum_{n=1}^{\infty} \frac{1}{5^n}$; $\to$ red $\sum_{n=1}^{\infty} \frac{1}{5^n}$ konvergira kao geometrijski red za koji je $q = \frac{1}{5}$ i $|q| < 1$ $\to$ pa konvergira i red $3 \cdot \sum_{n=1}^{\infty} \frac{1}{5^n}$, tj. red $\sum_{n=1}^{\infty} b_n$

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergira kao hiperharmonijski za koji je $p = 1$ $\to$ odavde slijedi da divergira i red $\sum_{n=1}^{\infty} \frac{2}{n}$, tj. red $\sum_{n=1}^{\infty} c_n$.

$$\left.\begin{array}{l}\text{Red } \sum_{n=1}^{\infty} b_n \text{ konv.} \\ \text{Red } \sum_{n=1}^{\infty} c_n \text{ diverg}\end{array}\right\} \to \text{red } \sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n}\right)$$

divergira

Redovi sa pozitivnim članovima

(113) Ispitati konvergenciju reda:

a) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

$\rightarrow a_n = \frac{1}{2n-1}$; Neka je $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0$

$\rightarrow$ Redovi suma $\sum_{n=1}^{\infty} b_n$ i $\sum_{n=1}^{\infty} a_n$ su ekvikonvergentni

Red $\sum_{n=1}^{\infty} \frac{1}{n}$ divergira kao hiperharmonijski red za koji je $p=1$ $\rightarrow$ pa divergira i red $\sum_{n=1}^{\infty} a_n$

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b) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$

$a_n = \frac{1}{n\sqrt{n+1}}$; Neka je $b_n = \frac{1}{n\sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n+1}}}{n\sqrt{n}} = 1 > 0 \rightarrow$ Redovi $\sum_{n=1}^{\infty} a_n$ i b_n su ekvikonvergentni

$\rightarrow$ Red $\sum_{n=1}^{\infty} b_n$ je konvergentan kao hiperharmonijski za koji je $p = \frac{3}{2} > 1$, pa konvergira i red $\sum_{n=1}^{\infty} a_n$

(114) Ispitati konvergenciju reda $\sum_{n=2}^{\infty} \frac{\sqrt{n+2}-\sqrt{n}}{n^\alpha}$

$a_n = \frac{\sqrt{n+2}-\sqrt{n}}{n^\alpha} \longrightarrow$ racionališemo

$a_n = \frac{n+2-n}{n^\alpha(\sqrt{n+2}+\sqrt{n})} = \frac{2}{n^\alpha(\sqrt{n+2}+\sqrt{n})}$

$\rightarrow$ neka je $b_n = \frac{2}{n^\alpha \sqrt{n}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{\cancel{2}}{\cancel{n^\alpha}(\sqrt{n+2}+\sqrt{n})}}{\frac{\cancel{2}}{\cancel{n^\alpha}\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}+\sqrt{n}} =$

$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{2}{n}}+1\right)} = \frac{1}{2}$

$0 < \lim_{n\to\infty} \frac{a_n}{b_n} < +\infty \longrightarrow \sum_{n=2}^{\infty} a_n$ i $\sum_{n=2}^{\infty} b_n$ su

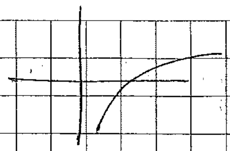
ekvikonvergentni

$\longrightarrow$ Red $\sum_{n=2}^{\infty} \frac{1}{n^{\alpha+1/2}}$ konvergira za $\alpha+\frac{1}{2}>1$

$\rightarrow$ Red $\sum_{n=2}^{\infty} b_n$ konv. za $\alpha > \frac{1}{2}$

$\rightarrow$ $\sum_{n=2}^{\infty} a_n$ konvergira za $\alpha > \frac{1}{2}$

(115.) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$



$\ln x > 1, \ \forall x > e$

$\forall n > 3$

$a_n = \frac{\ln n}{n}$

$\frac{1}{n} < \frac{\ln n}{n}, \ \forall n \geq 3$ $\longrightarrow$ Kriterijum upoređivanja

$b_n = \frac{1}{n}$

$b_n < a_n, \ \forall n \geq 3$

$\longrightarrow$ Red $\sum_{n=2}^{\infty} b_n$ divergira kao hiperharmonijski

za koji je $p=1$

1° $b_n < a_n, \ \forall n \geq 3$

2° $\sum_{n=2}^{\infty} b_n$ divergira

$\Big\}$ po poredbenom kriterijumu $\Longrightarrow \sum_{n=2}^{\infty} a_n$ divergira

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(116.) $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$

$a_n = \frac{1}{n \cdot 2^n}$ $\longrightarrow$ $\frac{1}{n \cdot 2^n} \leq \frac{1}{2^n}, \ \forall n \in \mathbb{N}$

$b_n = \frac{1}{2^n}$ $\longrightarrow$ Kriterijum upoređivanja

Red $\sum_{n=1}^{\infty} b_n$ je geom. red za koji je $q = \frac{1}{2}$, a kako je $|q| < 1$ $\longrightarrow$ to ovaj red konvergira

1° $a_n \leq b_n, \ \forall n \in \mathbb{N}$

2° $\sum_{n=1}^{\infty} b_n$ konv

$\Big\}$ $\Longrightarrow$ po kriterijumu upoređivanja ovaj red ($\sum_{n=1}^{\infty} a_n$ konv.)

(117.) $\sum_{n=1}^{\infty} \frac{1}{3^n+n}$ $\to$ $\frac{1}{3^n+n} \leq \frac{1}{3^n}, \forall n \in \mathbb{N}$

$a_n = \frac{1}{3^n+n}$

(118.) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2n}$ $\to$ $a_n = \frac{\sin 2n}{1+2^n} < \frac{1}{1+2^n} < \frac{1}{2^n}$

$\to$ Kriterijum upoređivanja

(119.) $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$ $\to$ Dalamberov kriterijum

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ $\to$ $<1 \to$ konvergira; $>1 \to$ divergira; $=1 \to$ neki drugi krit.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{2^{(n+1)^2}}}{\frac{(n!)^2}{2^{n^2}}} =$$

$$= \lim_{n\to\infty} \frac{\frac{(n+1)^2 \cdot \cancel{(n!)^2}}{\cancel{2^{n^2}} \cdot 2^{n+1}}}{\frac{\cancel{(n!)^2}}{\cancel{2^{n^2}}}} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} = 0 < 1$$

$\to$ brže raste exp. f-ja $\to$ Ψ

Red suma $\sum_{n=1}^{\infty} a_n$ konvergira po D. k.

(120.) $\frac{4}{2} + \frac{4\cdot 7}{2\cdot 6} + \frac{4\cdot 7\cdot 10}{2\cdot 6\cdot 10} + \ldots$ $\to$ Dalamberov kriterijum

$\to \sum_{n=1}^{\infty} \frac{4\cdot 7 \cdot \ldots \cdot (3n+1)}{2\cdot 6\cdot \ldots \cdot (4n-2)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{4\cdot 7 \cdot \ldots \cdot \cancel{(3n+1)}(3n+4)}{2\cdot 6 \cdot \ldots \cdot \cancel{(4n-2)}(4n+2)}}{\frac{4\cdot 7 \cdot \ldots \cdot \cancel{(3n+1)}}{2\cdot 6 \cdot \ldots \cdot \cancel{(4n-2)}}} = \cancel{3n}$$

$$= \lim_{n\to\infty} \frac{3n+4}{4n+2} = \frac{3}{4} < 1 \xrightarrow[D.K.]{} \sum_{n=1}^{\infty} a_n \text{ konv.}$$

(121.) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$; $a_n = \frac{n!}{n^n}$ $\longrightarrow$ Dalamberov kriterijum

$$\lim_{n\to\infty} \frac{a_{(n+1)}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} =$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-n} =$$

$$= \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^{-1} = e^{-1} = \frac{1}{e} < 1 \longrightarrow$$

$\longrightarrow$ po D.k $\longrightarrow$ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ konv.

(122.) $\sum_{n=1}^{\infty} \left(\frac{n-1}{n+1}\right)^{n(n-1)}$ $\longrightarrow$ Košijev kriterijum

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left(\frac{n-1}{n+1}\right)^{n-1} = \lim_{n\to\infty} \left(1+\frac{n-1}{n+1}-1\right)^{n-1} =$$

$$= \lim_{n\to\infty} \left(1+\frac{-2}{n+1}\right)^{\frac{n+1}{-2}\cdot\frac{-2}{n+1}(n-1)} =$$

$$= e^{\lim_{n\to\infty} \frac{-2n+2}{n+1}} = e^{-2} = \frac{1}{e^2} < 1$$

$\longrightarrow$ Po Koš. krit $\sum_{n=1}^{\infty} a_n$ konv.

(123.) $\sum_{n=1}^{\infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$ $\longrightarrow$ Košijev kriterijum

$$\frac{1+\cos n}{2+\cos n} = 1 - \frac{1}{2+\cos n} \leq 1 - \frac{1}{2+1} = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\lim_{n\to\infty} \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{3}\right)^{2n-\ln n}} \quad \longrightarrow$$

$$= \lim_{n\to\infty} \left(\frac{2}{3}\right)^{2-\frac{\ln n}{n}} = \frac{4}{9} < 1 \longrightarrow \text{na osnovu}$$

$\frac{\ln n}{n} \to 0$ → more Lopit. pr.

Koš. krit. red a_n. konv.

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(124) $\sum_{n=1}^{\infty} \left(\frac{n-1}{n}\right)^n$

$a_n = \left(\frac{n-1}{n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n-1}{n} = 1$ → ~~Koš~~ijev krit

$\longrightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^{-n(-1)} =$

$= e^{-1} = \frac{1}{e} \neq 0$

$\longrightarrow \lim_{n\to\infty} a_n \neq 0 \to \sum_{n=1}^{\infty} a_n$ ne konvergira

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~~(125) $\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\cdots(a+n)}$; $a>0$~~

~~Dalamberov krit~~

~~$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)\,n!}{(a+1)(a+2)\cdots(a+n)(a+n+1)}}{\frac{n!}{(a+1)(a+2)\cdots(a+n)}} =$~~

~~$= \lim_{n\to\infty}$~~

~~→ 125, 126, 127, 128, 129, 130, 131, 132, 133, 134~~

~~posle ~~Fura~~ ...~~ !

sl. str.

125. $\sum_{n=1}^{\infty} \frac{n!}{(a+1)(a+2)\cdots(a+n)}$; $a>0$

Dalamb.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)\cdot \cancel{n!}}{\cancel{(a+1)(a+2)\cdots(a+n)}(a+n+1)}}{\frac{\cancel{n!}}{\cancel{(a+1)(a+2)\cdots(a+n)}}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{a+n+1} = 1 \Rightarrow$$ Dalamberov kriterijum ne daje odgovor.

→ Primijenimo Rabov test → ako je $\lim > 1$ konv, $\lim < 1$ diverg.

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\cdot\left(\frac{a+n+1}{n+1} - 1\right) =$$

$$= \lim_{n\to\infty} n\cdot\frac{a}{n+1} = \lim_{n\to\infty} a\cdot\frac{n}{n+1} = a$$

Za $a>1$ → konvergira

Za $0<a<1$ red divergira

ako je $a=1$ – dobijamo brojni red

$$\sum_{n=1}^{\infty} \frac{n!}{2\cdot 3\cdot\ldots\cdot(1+n)} \rightarrow a_n = \frac{1}{n+1}$$

$b_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim \frac{n}{n+1} = 1 \Rightarrow$

$\Rightarrow$ $\sum a_n$ i $\sum b_n$ ekvikonvergentni

Red $\sum b_n$ div. kao hiperharm.

$\Rightarrow$ i $\sum a_n$ diverg.

126 $\sum_{n=1}^{\infty} \frac{(-2)^n + 3n^2}{3^n}$

$$\sum_{n=1}^{\infty} -\left(\frac{2}{3}\right)^n + \sum_{n=1}^{\infty} \frac{3n^2}{3^n}$$

a_n b_n

$\sum a_n$ je geom. red $|q| = \left|-\frac{2}{3}\right| < 1$

$\Rightarrow$ a_n konvergira 1°

$b_n = \frac{3n^2}{3^n}$ — Dalamberov krit

$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \frac{1}{3} < 1$ $\Rightarrow$ konvergira 2°

iz 1° i 2° $\Rightarrow$ red $\sum(a_n + b_n)$ konvergira

127. $\sum_{n=1}^{\infty} n \cdot e^{-n^2}$ Košijev integralni kriterijum

$a_n = n \cdot e^{-n^2}$

Neka je $f(x) = x \cdot e^{-x^2}$. Tada je $a_n = f(n), \forall n \in N$

1° $f(x) = \frac{x}{e^{x^2}}$; f je neprekidna kao kompozicija elementarnih u oblasti definisanosti, pa i na $[1, +\infty)$

2° $f(x) = \frac{x}{e^{x^2}} > 0, \ \forall x \in [1, +\infty)$

$x_1 < x_2$, $f(x_1) \geq f(x_2)$

3° $f'(x) = \frac{1-2x^2}{e^{x^2}} < 0 \iff 1 - 2x^2 < 0$

$\iff x \in \left(-\infty, -\frac{\sqrt{2}}{2}\right) \cup \left(\frac{\sqrt{2}}{2}, +\infty\right)$

sl. str.

Odavde slijedi da je $f'(x) < 0$ i za $\forall x \in [1, +\infty)$ pa je fja f na $[1, +\infty)$ opadajuća.

iz 1°, 2° i 3° na osnovu Košijevog integralnog kriterijuma slijedi da red $\sum_{n=1}^{\infty} a_n$ konvergira akko $\int_1^{+\infty} f(x)\,dx$ konvergira.

$$\int_1^{+\infty} f(x)\,dx = \lim_{B \to +\infty} \int_1^B x \cdot e^{-x^2}\,dx = \left| \begin{array}{l} \text{smjena } x^2 = t \\ 2x\,dx = dt \end{array} \right| = \dots =$$

$$= \lim_{B \to +\infty} \left(-\frac{1}{2} \cdot \underbrace{\frac{1}{e^{B^2}}}_{\to 0} + \frac{1}{2} \cdot \frac{1}{e} \right) = \lim_{n \to \infty} \frac{1}{2e} = \frac{1}{2e}$$

$\Rightarrow$ integral $\int_1^{+\infty} f(x)\,dx$ konv, pa i red $\sum a_n$ konv.

128

$$\sum_{n=2}^{\infty} \frac{1}{n \cdot \ln^2 n} \rightarrow a_n = \frac{1}{n^2 \cdot \ln^2 n}$$

Neka je $f(x) = \frac{1}{x \cdot \ln^2 x}$.

$f(n) = a_n$, $\forall n \geq 2$

1° neprekidna

$$f(x) = \frac{1}{x \cdot \ln^2 x} \Rightarrow \text{neprekidna na int. } [2, +\infty)$$

2° znak

$$f(x) = \frac{1}{\underbrace{x}_{>0} \cdot \underbrace{\ln^2 x}_{>0}} > 0, \quad \forall x \in [2, +\infty)$$

3° $f'(x) = -\frac{\ln x + 2}{x^2 \cdot \ln^3 x} < 0$, $\forall x \in [2, +\infty)$

$f \downarrow$ na $[2, +\infty)$

Iz 1°, 2° i 3° $\Rightarrow$ red $\sum_{n=2}^{\infty} a_n$ konv. akko $\int_2^{+\infty} f(x)\,dx$ konv.

$$\int_2^{+\infty} \frac{1}{x \cdot \ln^2 x}\,dx = \lim_{B \to +\infty} \int_2^{B} \frac{1}{x \cdot \ln^2 x}\,dx = \left| \ln x = t \right| = \ldots =$$

$$= \lim_{B \to +\infty} \left(-\frac{1}{\ln B} + \frac{1}{\ln 2} \right) = \frac{1}{\ln 2} \Rightarrow$$

$\Rightarrow$ ovaj integral konvergira $\Rightarrow$ red $\sum_{n=2}^{\infty} a_n$ konv.

znak se mijenja $\leftarrow$ **Alternativni redovi**

128

$$1 - \frac{3}{2} + \frac{5}{4} - \frac{7}{8}$$

* Samo Lajbnicov kriterijum

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2n-1}{2^{n-1}}$$

1° posmatramo niz $b_n = |a_n| = \frac{2n-1}{2^{n-1}}$

$$b_n - b_{n+1} = \frac{2n-1}{2^{n-1}} - \frac{2n+1}{2^n} = \frac{2(2n-1) - 2n - 1}{2^n} = \frac{2n-3}{2^n} > 0,$$

$\forall n \geq 2$

$\rightarrow$ (izbacujemo b_1, odnosno a_1, ali to ne utiče na konvergenciju.)

$b_n > b_{n+1}$, $\forall n \geq 2 \Rightarrow$ Niz (b_n) je opadajući

2° $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{2n-1}{2^{n-1}} = 0$;

Iz 1° i 2° na osnovu Lajbnicovog kriterijuma slijedi da red $\sum_{n=1}^{\infty} a_n$ konvergira.

130 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n!}{(2n)!}$

$$a_n = \frac{(-1)^n \cdot n!}{(2n)!} \qquad b_n = |a_n| = \frac{n!}{(2n)!}$$

$$1^\circ \quad \frac{b_n}{b_{n+1}} = \frac{\frac{\cancel{n!}}{\cancel{(2n)!}}}{\frac{(n+1)\cdot \cancel{n!}}{(2n+2)(2n+1)\cdot \cancel{(2n)!}}} = \frac{(2n+2)(2n+1)}{n+1} = \frac{4n^2+6n+2}{n+1} > 1$$

$*$

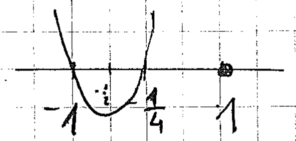
$$\frac{4n^2+6n+2}{n+1} > 1 \quad / \cdot (n+1) > 0, \ \forall n$$

$$4n^2+6n+2 > n+1$$

$$4n^2+5n+1 > 0$$

$$n_{1,2} = \frac{-5 \pm \sqrt{25-16}}{8} \Rightarrow n_{1,2} = \frac{-5 \pm 3}{8} \qquad \boxed{\begin{array}{l} n_1 = -1 \\ n_2 = -\frac{1}{4} \end{array}}$$

$$4n^2+5n+1 = 4(n+1)\left(n+\frac{1}{4}\right)$$



$$4n^2+5n+1 > 0, \ \forall n \in \mathbb{N}$$

$$\frac{b_n}{b_{n+1}} > 1 \Rightarrow b_n > b_{n+1} \Rightarrow$$

$\Rightarrow (b_n)$ je opadajući

$$2^\circ \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n!}{(2n)!} = \lim_{n\to\infty} \frac{n!}{1\cdot 2 \cdots n \cdot (n+1) \cdots (n+n)} =$$

$$= \lim_{n\to\infty} \frac{1}{n(n+1)\cdots(2n)} = 0$$

Iz 1° i 2° $\Rightarrow$ red a_n konv. po Lajbnicovom kriterijumu.

131 $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{2}{\sqrt{n^2+2}+n}$

$b_n = |a_n| = \frac{2}{\sqrt{n^2+2}+n}$

$n^2 < (n+1)^2 \Rightarrow n^2+2 < (n+1)^2+2$

$\sqrt{n^2+2}+n < \sqrt{(n+1)^2+2}+n$

$\Rightarrow \frac{2}{\sqrt{n^2+2}+n} > \frac{2}{\sqrt{(n+1)^2+2}+n} \Rightarrow$ red opada

$b_n > b_{n+1}, \forall n \in \mathbb{N}$

2° $\lim_{n\to\infty} b_n = 0$

Iz 1° i 2° $\Rightarrow$ red $\sum a_n$ konvergira

132. $\sum_{n=3}^{\infty} \frac{(-1)^{n-3} \cdot \sqrt{n}}{n+4}$

$b_n = |a_n| = \frac{\sqrt{n}}{n+4}$

1° $f(x) = \frac{\sqrt{x}}{x+4}$; $f'(x) = \frac{4-x}{2\sqrt{x}\cdot(x+4)^2} < 0 \Leftrightarrow 4-x<0$

$\Leftrightarrow x > 4$

$f'(x) < 0, \forall x \in [5, +\infty)$ f je opadajuća na $[5, +\infty)$;

$n < n+1$

$f(n) > f(n+1)$

$b_n > b_{n+1}, \forall n \geq 5$

2° $\lim_{n\to\infty} b_n = 0$

iz 1° i 2° $\Rightarrow$ red $\sum a_n$ konv